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String theory in β deformed spacetimes

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Abstract: Fluxbrane-like backgrounds obtained from flat space by a sequence of T-dualities and shifts of polar coordinates (β deformations) provide an interesting class of exactly solvable string theories. We compute the one-loop partition function for various such deformed spaces and study their spectrum of D-branes. For rational values of the B-field these models are equivalent to $\mathbb{Z}_N \times \mathbb{Z}_N$ orbifolds with discrete torsion. We also obtain an interesting new class of time-dependent backgrounds which resemble localized closed string tachyon condensation.

Keywords: D-branes, Superstring Vacua.

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1. Introduction and summary

String theory often requires modified notions of geometry. This is not only because the string world-sheet has a finite length scale, but also because a background for string theory is specified by NSNS and RR fluxes in addition to the metric. In this paper we study an interesting example of a closed string background with NSNS flux [1],

$$\begin{aligned} ds^2 &= dr_1^2 + dr_2^2 + \frac{r_1^2}{1+b^2r_1^2r_2^2}d\varphi_1^2 + \frac{r_2^2}{1+b^2r_1^2r_2^2}d\varphi_2^2\,, \\ e^{2(\phi-\phi_0)} &= \frac{1}{1+b^2r_1^2r_2^2}\,, \qquad B_{\varphi_1\varphi_2} = -\frac{br_1^2r_2^2}{1+b^2r_1^2r_2^2} \end{aligned} \tag{1.1}$$

and some generalizations thereof. This background shares some similarities with the Melvin fluxbrane solution [2–4]. In particular, like the Melvin case [5–7], string theory in (1.1) is exactly solvable [8]. In this case the solvability is due to the fact that (1.1) is actually equivalent to string theory on $\mathbb{R}^4$ after a T-duality, linear redefinition of the polar coordinates, followed by another T-duality. A sequence of such TsT dualities was recently used in [1] to find the supergravity solution dual to a certain marginal deformation of $\mathcal{N}=4$ Yang-Mills theory [9, 10] called the β deformation, but the procedure of using sequences of TsT transformations can of course be used more generally to find interesting new curved backgrounds which are solutions of the string equations of motion to all orders in α'.

When both radii r_i are large ($br_1r_2 \gg 1$), the metric (1.1) approaches

$$ds^2 \approx dr_1^2 + dr_2^2 + \frac{1}{b^2r_1^2}d\varphi_2^2 + \frac{1}{b^2r_2^2}d\varphi_1^2 = dr_1^2 + dr_2^2 + \beta^2 r_1^2 d\varphi_2'^2 + \beta^2 r_2^2 d\varphi_1'^2\,, \tag{1.2}$$

where we have performed T-duality $\varphi_i \to \varphi_i'$ on the shrinking φ_i circles and we have defined $\beta \equiv \alpha' b$. The form of the asymptotic metric (1.2) suggests that for rational β this background should be related to an orbifold of $\mathbb{C}^2$, and we will see below that this is indeed the case. In particular, the bosonic string in (1.1) at $\beta = k/N$ is equivalent to the orbifold $\mathbb{C}^2/\mathbb{Z}_N \times \mathbb{Z}_N$ with discrete torsion [11]. From this equivalence we can understand the effect of discrete torsion in a geometrical way by examining the solution (1.1). Intuitively, we can say that the discrete torsion smooths out the conical singularity otherwise present in the orbifold $\mathbb{C}^2/\mathbb{Z}_N \times \mathbb{Z}_N$.

Superstrings on (1.1) also give various orbifolds with discrete torsion, though the classification is more refined. We find that superstring theory on (1.1) is equivalent to

$$\begin{aligned} &\mathbb{C}^2/\mathbb{Z}_N \times \mathbb{Z}_N && \text{for} && \beta = 4\frac{k}{N}\,, \\ &\mathbb{C}^2/\mathbb{Z}_{2N} \times \mathbb{Z}_{2N} && \text{for} && \beta = \frac{(4k+2)}{N}\,, \\ &\mathbb{C}^2/\mathbb{Z}_{4N} \times \mathbb{Z}_{4N} && \text{for} && \beta = \frac{(4k+1)}{N} \quad \text{or} \quad \frac{(4k+3)}{N}\,. \end{aligned} \tag{1.3}$$

Superstring theory on such orbifolds has localized tachyons pinned to the orbifold points [12], and we will also see explicitly below that some states come from a type 0 GSO projection (i.e. there are (NS−,NS−) sectors), so there are also bulk tachyons as well. In fact, superstring theory on (1.1) has closed string tachyons for any value of β (not only rational)[1] and might therefore provide an interesting laboratory for studying closed string tachyon condensation, as the Melvin solution certainly has [4, 13].

The modified geometry implied by string theory can also be studied by using D-branes as probes. We study the spectrum of D-branes in the background (1.1), as has been done in the Melvin case in [14, 15]. In the orbifold limit when β is rational, the theory has both bulk D-branes and fractional D-branes, and we show agreement with results expected from the study of D-branes on orbifolds with discrete torsion [16–21].

[1] For example, in the notation of [8], states with $\nu_2 = 0$ and only zero-mode excitations have mass $\alpha' M^2 = -2\nu_1(J_{1R} - J_{1L})$, which can have either sign. We are grateful to A. Adams for pointing out the existence of such tachyonic states in the orbifold limit.

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We also study a generalization of (1.1) involving a three parameter deformation of $\mathbb{C}^3$ [1, 8, 22] which generically has tachyons but is supersymmetric when all three parameters are equal. Another generalization of (1.1) gives an interesting new time-dependent background which has some resemblence to the decay of localized closed string tachyons [12] (see [23] for a review and references therein).

The outline of this paper is as follows. We review the TsT transformation and the solution of the string sigma model for (1.1) in section 2. In section 3 we calculate, in both the operator and path-integral formalisms, the one-loop partition functions for bosonic and superstring theory in this background. We show that for rational β they agree with the various orbifolds with discrete torsion listed above. In section 4 we classify the various D-branes in (1.1) and compute boundary states for some of them, demonstrating agreement with expected results in the orbifold limits. In section 5 we summarize the conclusions of similar analysis for the three parameter deformation of $\mathbb{C}^3$, and finally in section 6 we comment on the time-dependent version of this background.

2. Solution of the closed string theory

In this section we review the solution of string theory in the background (1.1), following [8]. The sigma model describing closed strings in this background is

$$S = \frac{1}{2\pi\alpha'}\int d^2z\,\Big[\partial r_1\bar\partial r_1 + \partial r_2\bar\partial r_2 + \frac{r_1^2}{1+b^2r_1^2r_2^2}\partial\varphi_1\bar\partial\varphi_1 + \frac{r_2^2}{1+b^2r_1^2r_2^2}\partial\varphi_2\bar\partial\varphi_2 + \\ + \frac{br_1^2r_2^2}{1+b^2r_1^2r_2^2}(\partial\varphi_2\bar\partial\varphi_1 - \partial\varphi_1\bar\partial\varphi_2)\Big]\,. \tag{2.1}$$

We first T-dualize in $\varphi_1 \to \tilde\varphi_1$, finding

$$S = \frac{1}{2\pi\alpha'}\int d^2z\,\left[\partial r_1\bar\partial r_1 + \partial r_2\bar\partial r_2 + \frac{1}{r_1^2}\partial\tilde\varphi_1\bar\partial\tilde\varphi_1 + r_2^2(\partial\varphi_2 + b\partial\tilde\varphi_1)(\bar\partial\varphi_2 + b\bar\partial\tilde\varphi_1)\right]. \tag{2.2}$$

Then we define the new coordinate $\phi_2 = \varphi_2 + b\tilde\varphi_1$ and T-dualize again in $\tilde\varphi_1 \to \phi_1$. This gives

$$S = \frac{1}{2\pi\alpha'}\int d^2z\,\left[\partial r_1\bar\partial r_1 + \partial r_2\bar\partial r_2 + r_1^2\partial\phi_1\bar\partial\phi_1 + r_2^2\partial\phi_2\bar\partial\phi_2\right]. \tag{2.3}$$

Recognizing (2.3) as the sigma model for $\mathbb{C}^2$ written in polar coordinates, we define the complex free fields $X_1 = r_1e^{i\phi_1}$ and $X_2 = r_2e^{i\phi_2}$, whose action is just

$$S = \frac{1}{2\pi\alpha'}\int d^2z\,\left[\partial X_1\bar\partial\bar X_1 + \partial X_2\bar\partial\bar X_2\right]. \tag{2.4}$$

When two sigma models are related to each other by a T-duality in $X \to \tilde X$, their classical solutions are related by the formulas (see appendix A)

$$\begin{aligned}\partial\tilde X_\mu &= B_{\mu\nu}(X)\partial X^\nu - G_{\mu\nu}(X)\partial X^\nu\,,\\ \bar\partial\tilde X_\mu &= B_{\mu\nu}(X)\bar\partial X^\nu + G_{\mu\nu}(X)\bar\partial X^\nu\,.\end{aligned} \tag{2.5}$$

If we define $\tilde{\varphi}_2$ to satisfy $\phi_1 = \varphi_1 - b\tilde{\varphi}_2$, in analogy with the definition $\phi_2 = \varphi_2 + b\tilde{\varphi}_1$ we used above, then we find that the equations (2.5) imply that the relations

$$\begin{aligned} \partial\tilde{\varphi}_i &= -r_i^2\partial\phi_i = \frac{i}{2}[\bar{X}_i\partial X_i - X_i\partial\bar{X}_i]\,, \\ \bar{\partial}\tilde{\varphi}_i &= r_i^2\bar{\partial}\phi_i = -\frac{i}{2}[\bar{X}_i\bar{\partial}X_i - X_i\bar{\partial}\bar{X}_i] \end{aligned} \tag{2.6}$$

hold on-shell for $i = 1, 2$. Integrating the relation (2.6) over σ leads to the boundary condition

$$\tilde{\varphi}_i(t, \sigma + \pi) = \tilde{\varphi}_i(t, \sigma) - 2\pi\alpha'(J_i + \tilde{J}_i)\,, \qquad i = 1, 2, \tag{2.7}$$

where we have introduced the familiar left- and right-moving angular momentum operators

$$\begin{aligned} J_i &= \frac{i}{4\pi\alpha'}\int_0^\pi d\sigma[X_i\partial\bar{X}_i - \bar{X}_i\partial X_i]\,, \\ \tilde{J}_i &= \frac{i}{4\pi\alpha'}\int_0^\pi d\sigma[X_i\bar{\partial}\bar{X}_i - \bar{X}_i\bar{\partial}X_i]\,, \qquad i = 1, 2\,. \end{aligned} \tag{2.8}$$

From (2.6) it follows that closed string boundary conditions

$$\varphi_i(t, \sigma + \pi) = \varphi_i(t, \sigma)\,, \qquad i = 1, 2 \tag{2.9}$$

for the original sigma model (2.1) translate into

$$\begin{aligned} \phi_1(t, \sigma + \pi) &= \phi_1(t, \sigma) + 2\pi\beta(J_2 + \tilde{J}_2)\,, \\ \phi_2(t, \sigma + \pi) &= \phi_2(t, \sigma) - 2\pi\beta(J_1 + \tilde{J}_1)\,. \end{aligned} \tag{2.10}$$

We conclude from this analysis that the sigma model (2.1) reduces to two free complex bosons with the twisted boundary conditions

$$\begin{aligned} X_1(t, \sigma + \pi) &= e^{2\pi i\beta(J_2 + \tilde{J}_2)}X_1(t, \sigma)\,, \\ X_2(t, \sigma + \pi) &= e^{-2\pi i\beta(J_1 + \tilde{J}_1)}X_2(t, \sigma)\,. \end{aligned} \tag{2.11}$$

Note that since the angular momenta J_i are integer quantized, it is manifest from (2.11) that the theory is periodic under $\beta \to \beta + 1$ in the bosonic string case. We can also consider the supersymmetric completion of (2.1), in which case the periodicity becomes $\beta \sim \beta + 2$, as we will see explicitly below.

3. One-loop partition functions

The one-loop vacuum amplitude in closed string theory can be expressed as

$$Z_{T^2} = iV_{D-4}\int\frac{d\tau d\bar{\tau}}{4\tau_2}\frac{1}{(4\pi^2\alpha'\tau_2)^{\frac{1}{2}(D-4)}}Z(\tau, \bar{\tau}, \beta)\,, \tag{3.1}$$

where V_{D-4} is the volume of the non-compact dimensions described by a trivial CFT which we omit in the discussion below, and $Z(\tau, \bar{\tau}, \beta)$ is the 4D partition function which we are interested in. We take $D = 10$ for the superstring and $D = 26$ for the bosonic string.

3.1 Operator formalism

The partition function Z can be computed in the operator formalism as

$$Z(\tau,\bar\tau,\beta) = \mathrm{Tr}\left[q^{L_0}\bar q^{\tilde L_0}\right], \tag{3.2}$$

with $q = e^{2\pi i\tau}$ and $\tau = \tau_1 + i\tau_2$. In [8] it was shown that the canonical quantization of the model (2.1) leads to the expressions

$$\begin{aligned} L_0 &= N - a - (\nu_1 - [\nu_1])J_1 - (\nu_2 - [\nu_2])J_2\,, \\ \tilde L_0 &= \tilde N - \tilde a + (\nu_1 - [\nu_1])\tilde J_1 + (\nu_2 - [\nu_2])\tilde J_2\,, \end{aligned} \tag{3.3}$$

where a and $\tilde a$ are the usual normal-ordering constants, $[x]$ denotes the greatest integer less than or equal to x, and

$$\nu_1 = \beta(J_2 + \tilde J_2)\,, \qquad \nu_2 = -\beta(J_1 + \tilde J_1)\,. \tag{3.4}$$

In order to calculate the trace (3.2) we find it convenient to follow a similar calculation performed in [5, 6, 24]. We begin by inserting delta-functions to write

$$Z = \int d^2 j_1 d^2 j_2 \;\; \mathrm{Tr}[q^{L_0}\bar q^{\tilde L_0}\delta^2(J_1 - j_1)\delta^2(J_2 - j_2)]\,. \tag{3.5}$$

These enable us to set $J_i = j_i$ and $\tilde J_i = \bar\jmath_i$ inside the trace. With the help of the formula

$$\delta^2(z) = \int d^2\chi\; e^{2\pi i\chi z + 2\pi i\bar\chi\bar z} \tag{3.6}$$

we then obtain an expression in which the left- and right-moving traces factorize,

$$\begin{aligned} Z \,=& \int d^2 j_1 d^2 j_2 d^2\chi_1 d^2\chi_2\; q^{-(\nu_1-[\nu_1])j_1-(\nu_2-[\nu_2])j_2}\bar q^{(\nu_1-[\nu_1])\bar\jmath_1+(\nu_2-[\nu_2])\bar\jmath_2} \times \\ &\times e^{-2\pi i(\chi_1 j_1+\bar\chi_1\bar\jmath_1+\chi_2 j_2+\bar\chi_2\bar\jmath_2)}\,\mathrm{Tr}\left[q^{N-a}e^{2\pi i\chi_1 J_1+2\pi i\chi_2 J_2}\right]\mathrm{Tr}\left[\bar q^{\tilde N-\tilde a}e^{2\pi i\bar\chi_1\tilde J_1+2\pi i\bar\chi_2\tilde J_2}\right]. \end{aligned} \tag{3.7}$$

The result (3.7) is equally valid for the bosonic string and the superstring. Let us consider first the superstring, in which case the oscillator traces lead to (see appendix B)

$$F(\chi_1,\chi_2,\tau) \equiv \left|\frac{\vartheta_1(\frac12(\chi_1+\chi_2)|\tau)^2\vartheta_1(\frac12(\chi_1-\chi_2)|\tau)^2}{\eta(\tau)^6\vartheta_1(\chi_1|\tau)\vartheta_1(\chi_2|\tau)}\right|^2. \tag{3.8}$$

This result is correct for the standard type II GSO projection. As shown in [8], when $[\nu_1]+[\nu_2]$ is odd one must use the opposite GSO projection; we will incorporate this fact momentarily. Even though the partition function (3.8) is expressed in the form which emerges naturally from the Green-Schwarz formalism, it is equivalent to the more familiar expression in the NSR formalism via Jacobi's identity (see appendix B).

In (3.7) we can perform the change of variables

$$\chi_i \to \chi_i + \tau[\nu_i], \qquad \bar\chi_i \to \bar\chi_i + \bar\tau[\nu_i] \tag{3.9}$$

to eliminate the appearance of $[\nu]_i$. The factor (3.8) is invariant under this shift if $[\nu_1]+[\nu_2]$ is even, but for $[\nu_1]+[\nu_2]$ odd we have

$$F(\chi_1+[\nu_1]\tau,\chi_2+[\nu_2]\tau,\tau)=\left|\frac{\vartheta_4(\frac{1}{2}(\chi_1+\chi_2)|\tau)^2\vartheta_4(\frac{1}{2}(\chi_1-\chi_2)|\tau)^2}{\eta(\tau)^6\vartheta_1(\chi_1|\tau)\vartheta_1(\chi_2|\tau)}\right|^2, \tag{3.10}$$

which is precisely the required oscillator trace for the opposite GSO projection. Finally we can perform the j_i integrals to arrive at

$$Z=\int\frac{d^2\chi_1 d^2\chi_2}{(2\beta\tau_2)^2}e^{\frac{\pi}{\tau_2\beta}(\chi_1\bar\chi_2-\chi_2\bar\chi_1)}\left|\frac{\vartheta_1(\frac{1}{2}(\chi_1+\chi_2)|\tau)^2\vartheta_1(\frac{1}{2}(\chi_1-\chi_2)|\tau)^2}{\eta(\tau)^6\vartheta_1(\chi_1|\tau)\vartheta_1(\chi_2|\tau)}\right|^2, \tag{3.11}$$

which naturally incorporates the proper GSO projections. This result for the one-loop amplitude can also be obtained by performing the path integral, as we will show in the following subsection. Plugging (3.11) into (3.1) gives a modular invariant amplitude in $D=10$.

The result (3.11) is actually periodic in β,

$$Z(\tau,\bar\tau,\beta+2)=Z(\tau,\bar\tau,\beta)\,. \tag{3.12}$$

In order to expose this symmetry, we use the fact that the combination of ϑ-functions appearing in (3.11) is invariant under

$$\chi_1\to\chi_1+2(m'-m\tau)\,,\qquad \chi_2\to\chi_2+2(n'-n\tau)\,, \tag{3.13}$$

for $m,m',n,n'\in\mathbb{Z}$. We can exploit this periodicity to divide the integral over χ_2 into a sum over n and n' as well as an integral over the torus $(0,0)\sim(2,2\tau)$,

$$Z=\sum_{n,n'\in\mathbb{Z}}\int\frac{d^2\chi_1}{2\beta\tau_2}\int_{T^2}\frac{d^2\chi_2}{2\beta\tau_2}e^{\frac{\pi}{\tau_2\beta}(\chi_1\bar\chi_2-\chi_2\bar\chi_1)}e^{\frac{2\pi}{\tau_2\beta}[\chi_1(n'-n\bar\tau)-\bar\chi_1(n'-n\tau)]}F(\chi_1,\chi_2,\tau)\,. \tag{3.14}$$

The sum over n and n' gives the delta-functions

$$\frac{\tau_2\beta^2}{2}\sum_{p,p'\in\mathbb{Z}}\delta^2(\chi_1-\frac{1}{2}\beta(p'-p\tau)) \tag{3.15}$$

which allow easy evaluation of the χ_1 integral,

$$Z=\frac{1}{8\tau_2}\sum_{p,p'\in\mathbb{Z}}\int_{T^2}d^2\chi_2\;e^{\frac{\pi}{2\tau_2}[(p'-p\tau)\bar\chi_2-(p'-p\bar\tau)\chi_2]}F(\frac{1}{2}\beta(p'-p\tau),\chi_2,\tau)\,. \tag{3.16}$$

In this form the periodicity under $\beta\to\beta+4$ is manifest, and the finer periodicity (3.12) can be seen by shifting the χ_2 integration variable by $\chi_2\to\chi_2+(p'-p\tau)$.

3.2 Path-integral formalism

The torus amplitude (3.11) may also be obtained by directly evaluating the one-loop functional determinant of the sigma model action (2.1). This calculation is facilitated by the introduction of a useful choice of auxiliary fields, as in a similar calculation for the Melvin background [5, 6]. In our case we introduce two complex auxiliary fields U and V with the action

$$S = \frac{1}{2\pi\alpha'} \int d^2z \Big[\partial r_1 \bar{\partial} r_1 + \partial r_2 \bar{\partial} r_2 + r_2^2 \partial\varphi_2 \bar{\partial}\varphi_2 + b^2 r_2^2 V\bar{V} + r_1^2 U\bar{U} + \\ + \bar{V}(\partial\varphi_1 + b r_2^2 \partial\varphi_2) - V(\bar{\partial}\varphi_1 - b r_2^2 \bar{\partial}\varphi_2) - U\bar{V} + \bar{U}V \Big] . \qquad (3.17)$$

It is straightforward to check that integrating out U and V recovers (2.1).

If we instead start from (3.17) and perform the path integral over φ_1, we find the flatness condition $\partial\bar{V} - \bar{\partial}V = 0$ which is solved by writing

$$V = v + \partial\varphi \,, \qquad \bar{V} = \bar{v} + \bar{\partial}\varphi \qquad (3.18)$$

in terms of a complex constant (zero-mode) v and a new field φ. Actually, the flatness condition arises from integrating out only the non-zero modes of φ_1. There is a term in the action,

$$\int (\bar{v}\partial\varphi_1 - v\bar{\partial}\varphi_1) \qquad (3.19)$$

which depends also on the zero-mode of φ_1, which is given by

$$\varphi_1 = m\sigma_1 + \frac{m' - m\tau_1}{\tau_2}\sigma_2 \,. \qquad (3.20)$$

We will incorporate this zero-mode shortly.

The path integral over V therefore reduces to an ordinary integral over v and a path integral over φ, although note that the latter has no zero mode since only derivatives of φ appear in (3.18). We obtain

$$S = \frac{1}{2\pi\alpha'} \int d^2z \Big[\partial r_1 \bar{\partial} r_1 + r_1^2 U\bar{U} - U(\bar{v} + \bar{\partial}\varphi) + \bar{U}(v + \partial\varphi) + \\ + \partial r_2 \bar{\partial} r_2 + r_2^2 (bv + b\partial\varphi + \partial\varphi_2)(b\bar{v} + b\bar{\partial}\varphi + \bar{\partial}\varphi_2) \Big] . \qquad (3.21)$$

Next we change variables in the path integral from (φ, φ_2) to $(\varphi, \varphi_2 + b\varphi)$. Then we can integrate out φ (which, recall, has no zero modes, so we don't have to worry about total derivative terms such as (3.19)) to obtain another flatness condition $\partial\bar{U} - \bar{\partial}U = 0$. Again we can write

$$U = u + \partial\theta, \quad \bar{U} = \bar{u} + \bar{\partial}\theta \,, \qquad (3.22)$$

in terms of a constant u and a function θ with no zero-mode. Finally we arrive at the result

$$S = \frac{1}{2\pi\alpha'} \int dz^2 \Big[\partial r_1 \bar{\partial} r_1 + r_1^2 (u + \partial\theta)(\bar{u} + \bar{\partial}\theta) - u\bar{v} + \bar{u}v - \bar{v}\partial\theta + v\bar{\partial}\theta + \\ + \partial r_2 \bar{\partial} r_2 + r_2^2 (bv + b\partial\varphi + \partial\varphi_2)(b\bar{v} + b\bar{\partial}\varphi + \bar{\partial}\varphi_2) \Big] . \qquad (3.23)$$

We can set the terms $\int(-\bar{v}\partial\theta + v\bar{\partial}\theta)$ to zero because θ has no zero-modes, and then rewrite this action in terms of the coordinates $Z_1 = r_1 e^{i\theta}$ and $Z_2 = r_2 e^{i(\varphi_2 + b\varphi)}$ as

$$S = \frac{1}{2\pi\alpha'}\int d^2z \Big[\frac{1}{2}\left[(\partial + iu)Z_1\cdot(\bar{\partial} - i\bar{u})\bar{Z}_1 + (\bar{\partial} + i\bar{u})Z_1\cdot(\partial - iu)\bar{Z}_1\right] + \\ + \frac{1}{2}\left[(\partial + iv)Z_2\cdot(\bar{\partial} - i\bar{v})\bar{Z}_2 + (\bar{\partial} + i\bar{v})Z_2\cdot(\partial - iv)\bar{Z}_2\right] + \frac{1}{b}(\bar{u}v - \bar{v}u)\Big]. \tag{3.24}$$

We still have not yet performed the path integral over the zero-modes (3.20) of φ_1, but if we now combine those zero-modes together with the non-zero modes of θ into a conventional field, then we can regard $Z_1 = r_1 e^{i\theta}$ as a completely ordinary free field.

The final form (3.24) is quadratic, so the one-loop functional determinant is easily evaluated using the methods of [5−7, 25]. The result agrees with (3.11) (when the appropriate Green-Schwarz fermions are added to (3.24)). The integrals over the zero modes u, v identified with the integrals over the auxiliary parameters χ_1, χ_2 in (3.11).

3.3 Relation to $\mathbb{C}^2/G$ orbifolds

We will now show that the result (3.16) for the partition function reduces to various orbifolds of $\mathbb{C}^2$ with discrete torsion when β is rational. In general, orbifolds with discrete torsion are defined by adding extra phases ϵ in the sum over twisted sectors in the partition function [11]. The one-loop amplitude for a theory with abelian orbifold group G can be written as

$$Z = \frac{1}{|G|}\sum_{g,h\in G}\epsilon(g,h)Z_{g,h}\,. \tag{3.25}$$

Modular invariance requires that $\epsilon(g,h)$ should satisfy the relations $\epsilon(g,hk) = \epsilon(g,h)\epsilon(g,k)$, $\epsilon(g,h) = \epsilon(h,g)^{-1}$ and $\epsilon(g,g) = 1$ [11].

Let us first consider the case $\beta = 4/N$ such that N and 4 are coprime. Generalizing (3.14), we can use the double periodicity (3.13) to divide both χ_i integrals,

$$Z = \sum_{m,m',n,n'\in\mathbb{Z}}\int_{T^2}\frac{d^2\chi_1}{2\beta\tau_2}\int_{T^2}\frac{d^2\chi_2}{2\beta\tau_2}e^{\frac{\pi}{\tau_2\beta}(\chi_1\bar{\chi}_2 - \chi_2\bar{\chi}_1)}e^{\frac{8\pi i}{\beta}(m'n - mn')}\times \\ \times e^{\frac{2\pi}{\tau_2\beta}[\chi_1(n'-n\bar{\tau}) - \bar{\chi}_1(n'-n\tau)]}e^{\frac{2\pi}{\tau_2\beta}[\chi_2(m'-m\bar{\tau}) - \bar{\chi}_2(m'-m\tau)]}F(\chi_1,\chi_2,\tau)\,. \tag{3.26}$$

The phase $e^{\frac{8\pi i}{\beta}(m'n - mn')}$ drops out when $\beta = 4/N$, and we can perform the summations to obtain the delta-functions

$$\frac{1}{4}(\tau_2\beta^2)^2\sum_{p,p',q,q'\in\mathbb{Z}}\delta^2(\chi_1 - \frac{1}{2}\beta(p' - p\tau))\delta^2(\chi_2 - \frac{1}{2}\beta(q' - q\tau))\,. \tag{3.27}$$

We see that the values of χ_1 and χ_2 selected by the delta-functions (3.27) are those with $0 \le p, p', q, q' \le N-1$. In this case the partition function becomes

$$Z = \frac{1}{N^2}\sum_{p,p',q,q'=0}^{N-1}e^{\frac{2\pi i}{N}(p'q - pq')}\left|\frac{\vartheta_1(\frac{1}{2}(\nu_{p'p} + \nu_{q'q})|\tau)^2\vartheta_1(\frac{1}{2}(\nu_{p'p} - \nu_{q'q})|\tau)^2}{\eta(\tau)^6\vartheta_1(\nu_{p'p}|\tau)\vartheta_1(\nu_{q'q}|\tau)}\right|^2, \tag{3.28}$$

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where we have defined $\nu_{m'm} = \frac{2}{N}(m' - m\tau)$. We recognize (3.28) as the usual orbifold partition function for $\mathbb{C}^2/\mathbb{Z}_N \times \mathbb{Z}_N$ defined by the two $\mathbb{Z}_N$ actions

$$g_1 : (X_1, X_2) \to (e^{4\pi i/N} X_1, X_2)\,, \qquad g_2 : (X_1, X_2) \to (X_1, e^{4\pi i/N} X_2)\,, \tag{3.29}$$

multiplied by an extra phase factor $e^{\frac{2\pi i}{N}(p'q - pq')}$ which indicates the presence of discrete torsion.

For $G = \mathbb{Z}_N \times \mathbb{Z}_N$, the possible phases which satisfy the consistency conditions listed below (3.25) are given by

$$\epsilon((m,l),(m',l')) = e^{\frac{2\pi i}{N}(ml' - m'l)n}\,, \tag{3.30}$$

where (m, l) denotes a boundary condition twisted by $(g_1)^m(g_2)^l$. We have $N - 1$ independent choices for the discrete torsion phase $n = 1, 2, 3, \ldots, N - 1$ ($n = 0$ corresponds to having no discrete torsion). Our result (3.28) for $\beta = 4/N$ corresponds to the particular case $n = 1$.

Similar results hold for more general rational values of β, though we will only sketch the derivations, which are elementary. There are four cases to consider, $\beta = \frac{4k+l}{N}$ for $l = 0, 1, 2, 3$. For these general cases it is convenient to start with the formula (3.16) instead of (3.26). For example, for $\beta = 4k/N$ (i.e. $l = 0$) we can decompose the summation variable $p = Nq + r$ (and p' similarly) to write the sum over p as

$$\sum_{p\in\mathbb{Z}} = \sum_{q\in\mathbb{Z}} \sum_{r=0}^{N-1}\,. \tag{3.31}$$

The ϑ-functions in (3.16) are then independent of q and q'. Summing over these variables gives delta-functions similar to (3.27) which localize the χ_2 integral and give precisely the result (3.28).

For $\beta = \frac{4k+2}{N}$ we must use the trick (3.31) with $p = 2Nq + r$ (and let r run from 0 to $2N - 1$) in order to decouple the sum over q, and for $\beta = \frac{4k+1}{N}$ or $\frac{4k+3}{N}$ we must use $p = 4Nq + r$, with r running from 0 to $4N - 1$. At the end of the day, we find that the four cases $l = 0, 1, 2, 3$ are respectively equivalent to the orbifolds $\mathbb{C}^2/\mathbb{Z}_N \times \mathbb{Z}_N$, $\mathbb{C}^2/\mathbb{Z}_{4N} \times \mathbb{Z}_{4N}$, $\mathbb{C}^2/\mathbb{Z}_{2N} \times \mathbb{Z}_{2N}$ and $\mathbb{C}^2/\mathbb{Z}_{4N} \times \mathbb{Z}_{4N}$ with discrete torsion, as summarized in (1.3). The discrete torsion in these four cases is given by (3.30) with n defined by $kn = 1 \bmod N$, $(4k + 1)n = 1 \bmod 4N$, $(2k + 1)n = 1 \bmod 2N$, or $(4k + 1)n = 1 \bmod 4N$, respectively. They are non-supersymmetric orbifolds in type II string theory defined by the actions of a $\beta\pi$ rotation in each complex plane.

At the values $\beta = \frac{4k+1}{N}, \frac{4k+2}{N}, \frac{4k+3}{N}$ they can simultaneously be viewed as orbifolds with discrete torsion in type 0 string theory. This is because in these cases, the partition function includes sectors with the (NS−,NS−) GSO projection. For example, when $\beta = 2/N$ we find

$$Z = \frac{1}{4N^2} \sum_{l,l',n,n'=0}^{2N-1} e^{\frac{\pi i}{N}(l'n - ln')} \left| \frac{\vartheta_1(\frac{1}{4}(\nu_{l'l} + \nu_{n'n})|\tau)^2 \vartheta_1(\frac{1}{4}(\nu_{l'l} - \nu_{n'n})|\tau)}{\eta(\tau)^6 \vartheta_1(\frac{1}{2}\nu_{l'l}|\tau)\vartheta_1(\frac{1}{2}\nu_{n'n}|\tau)} \right|^2 . \tag{3.32}$$

The terms in the sum with $(l,n) = (N,0)$ or $(l,n) = (0,N)$ correspond to the (NS−,NS−) GSO projection (this can be seen by using Jacobi's abstruse identity). These sectors contain the usual bulk closed string tachyons of type 0 string theory.

3.4 Bosonic string

We can also consider the bosonic string in the sigma model (2.1) with $D-4 = 22$ additional flat spacetime dimensions. The only subtlety in calculating the partition function in the operator formalism is the nontrivial zero-point energy. We find

$$a = \tilde{a} = 1 + \frac{1}{2}(\nu_1 - [\nu_1])^2 + \frac{1}{2}(\nu_2 - [\nu_2])^2 \,. \tag{3.33}$$

After a calculation similar to that in [5, 6, 24] we find the partition function

$$Z = \int \frac{d^2\chi_1\, d^2\chi_2}{(2\beta\tau_2)^2} e^{\frac{\pi}{\beta\tau_2}(\chi_1\bar{\chi}_2 - \chi_2\bar{\chi}_1)} e^{-\frac{\pi}{2\tau_2}(\chi_1-\bar{\chi}_1)^2} e^{-\frac{\pi}{2\tau_2}(\chi_2-\bar{\chi}_2)^2} \left| \frac{1}{\eta(\tau)^{18}\vartheta_1(\chi_1|\tau)\vartheta_2(\chi_2|\tau)} \right|^2 . \tag{3.34}$$

Using manipulations similar to those above it is easy to show that this partition function is invariant under

$$\beta \to \beta + 1 \,, \tag{3.35}$$

as expected. In the case of $\beta = k/N$, we always have the the orbifold $\mathbb{C}^2/\mathbb{Z}_N \times \mathbb{Z}_N$ with discrete torsion.

4. D-branes

In general, D-branes are important probes of geometrical aspects of spacetime in string theory. Also they often offer us various interesting models of Yang-Mills theory. In this section we study D-branes in the background (1.1). To define D-branes, it is useful to analyze them in the free field representation (2.4), (2.11), where we can find an important class[2] of boundary states by imposing Dirichlet or Neumann boundary conditions on these free fields $X^1 = r_1 e^{i\phi_1}$ and $X^2 = r_2 e^{i\phi_2}$.

We can consider the following nine possibilities for the boundary conditions of (r_1, ϕ_1) and (r_2, ϕ_2):

$$\begin{array}{lll} \text{(a)} : (D,D),(D,D)\,, & \text{(b)} : (N,N),(N,N)\,, & \text{(c)} : (D,D),(N,N)\,, \\ \text{(d)} : (D,D),(N,D)\,, & \text{(e)} : (N,D),(N,D)\,, & \text{(f)} : (N,D),(N,N)\,, \\ \text{(c}'\text{)} : (N,N),(D,D)\,, & \text{(d}'\text{)} : (N,D),(D,D)\,, & \text{(f}'\text{)} : (N,N),(N,D)\,. \end{array} \tag{4.1}$$

Notice that $(c'), (d'), (f')$ are essentially the same as $(c), (d), (f)$ so we will not mention them.

[2] It is possible that there exist more general D-brane boundary states which cannot be obtained from this method. One of such examples may be a D0-brane which is oscillating around the origin due to the $r_{1,2}$ dependent dilaton.

4.1 Toroidal D2-brane and quantization of b

First we discuss the (a)-type D-brane in (4.1), defined by imposing the Dirichlet boundary condition in all directions $(X_1, \bar{X}_1, X_2, \bar{X}_2)$, i.e.

$$(\partial - \bar{\partial})\phi_{1,2} = (\partial - \bar{\partial})r_{1,2} = 0\,. \tag{4.2}$$

We can rewrite (4.2) in terms of the original sigma model coordinates φ_1 and φ_2 using the formula (A.3) given in the appendix A. We find

$$\begin{aligned}
(\partial - \bar{\partial})\phi_1 &= \frac{br_2^2}{1+b^2r_1^2r_2^2}(\partial + \bar{\partial})\varphi_2 + \frac{1}{1+b^2r_1^2r_2^2}(\partial - \bar{\partial})\varphi_1 = 0\,,\\
(\partial - \bar{\partial})\phi_2 &= -\frac{br_1^2}{1+b^2r_1^2r_2^2}(\partial + \bar{\partial})\varphi_1 + \frac{1}{1+b^2r_1^2r_2^2}(\partial - \bar{\partial})\varphi_2 = 0\,.
\end{aligned} \tag{4.3}$$

Comparing (4.3) with the standard expression for the (mixed) Neumann boundary condition

$$G_{\mu\nu}(\partial + \bar{\partial})X^\nu + (B_{\mu\nu} + 2\pi\alpha' F_{\mu\nu})(\bar{\partial} - \partial)X^\nu = 0\,, \tag{4.4}$$

we see that this D-brane represents a D2-brane whose world-volume is a torus $0 \leq \varphi_{1,2} \leq 2\pi$ at fixed values of r_1 and r_2. Moreover we find from (4.4) that there is a non-zero gauge flux F on this toroidal brane (these computations are very similar to those in the Melvin background [26])

$$2\pi\alpha' F_{\varphi_1\varphi_2} = \frac{1}{b}\,. \tag{4.5}$$

This D2-brane is stabilized by the presence of this flux, as we will see shortly, even though it wraps a topologically trivial cycle in spacetime. Flux quantization requires

$$\frac{1}{2\pi}\,\mathrm{Tr}\int d\varphi_1 d\varphi_2\ F_{\varphi_1\varphi_2} = \frac{k}{\beta} \in \mathbb{Z}\,, \tag{4.6}$$

where k is the number of the D2-branes. Thus such a D-brane system is allowed when we have[3]

$$\beta = \frac{k}{N}\,, \tag{4.7}$$

for an integer N.

When β is irrational and r_1 and r_2 are non-zero this configuration is not consistent with the quantization and thus does not exist. Still such a D-brane can exist when $r_1 = 0$ or $r_2 = 0$. However, its world-volume shrinks to zero size and it should be called a D0-brane instead of a D2-brane, as is clear from (4.3).

[3] In this analysis we do not see the modulo 4 distinction as in (1.3). This distinction comes from the different spin structures of the superstring, but our D0-brane analysis is only sensitive to the bosonic degrees of freedom. Indeed, in the bosonic string, as we have seen in section 3.4, there is no distinction between the four cases.

This phenomenon can also be found from the effective theory analysis. Consider k D2-branes with magnetic flux $f = 2\pi\alpha' F_{\varphi_1\varphi_2}$ wrapped on a torus $0 \le \varphi_1, \varphi_2 \le 2\pi$ for fixed values of (r_1, r_2) in the background (1.1). We can find its energy from the DBI action

$$\begin{aligned} M_{N,f} &= \frac{e^{-\phi}}{4\pi^2(\alpha')^{3/2}} \mathrm{Tr} \int d\varphi_1 d\varphi_2 \sqrt{\det(G + B + 2\pi\alpha' F)} \\ &= k \frac{e^{-\phi_0}}{\alpha'^{3/2}} \sqrt{r_1^2 r_2^2 (fb-1)^2 + f^2}\,. \end{aligned} \tag{4.8}$$

For generic values of f, the energy of the D2-branes is only stabilized at $r_1 r_2 = 0$, where it should be regarded as a D0-brane. In the rational case $f = 1/b = \alpha' N/k$, however, there exist stable D2-brane configurations for any non-zero values of r_1 and r_2. These values precisely agree with (4.5) and should be quantized according to (4.7). Then the system can be regarded as a bound state of N D0-branes and k D2-branes. Due to the magnetic flux, the open string theory becomes non-commutative. Following the standard formula [27]

$$\theta^{ij} = 2\pi\alpha' (G + B + 2\pi\alpha' F)^{-1\ ij}_{\text{asymmetric}} \tag{4.9}$$

we can see that the world-volume of the D2-brane becomes a non-commutative torus with rational non-commutativity parameter[4] $2\pi\beta = 2\pi k/N$, i.e. a fuzzy torus. On the other hand, the backgrounds with irrational values of β can be regarded as an $N \to \infty$ limit, in which case the mass of D2-brane becomes infinite. These correspond to a non-commutative torus with irrational non-commutativity.

In the rational case, we have seen the model (2.1) becomes equivalent to the orbifold $\mathbb{C}^2/\mathbb{Z}_N \times \mathbb{Z}_N$ with discrete torsion. In this model, following general arguments [16, 17] (see also appendix C), we have two kinds of D0-branes. One is a regular bulk D0-brane, which has moduli to freely move around the four-dimensional space. The other is called a fractional D0-brane, which is stuck on the fixed lines $r_1 r_2 = 0$ and whose mass is $1/N$ times smaller than that of a bulk D0-brane. It is now clear that the toroidal D2-brane and the localized D0-brane that we find in the background (1.1) are equivalent to the bulk D0-brane and fractional D0-brane, respectively. Another check of this fact is that the mass (4.8) of the D2-brane in the rational case becomes

$$M_{N,f} = N \frac{e^{-\phi_0}}{\sqrt{\alpha'}}\,, \tag{4.10}$$

which is precisely the same as that of N D0-branes on the fixed lines.

We would also like to point out that the properties of a fractional D0-brane in orbifolds with discrete torsion are actually not the same as those in ordinary orbifolds without discrete torsion. In the latter case, the brane cannot move away from the fixed points (or lines) at all even if we push it by some external force. On the other hand, in the former case, it can be pushed off the fixed points by adding extra force. This shows that the fixed points (or lines) are not rigid in the presence of discrete torsion. This fact is also clear from the geometry (1.1), in which the geometry around $r_1 r_2 = 0$ is smoothed out.

[4] We define the non-commutative torus with the non-commutativity $2\pi\beta$ by the algebra $UV = VUe^{2\pi i\beta}$.

4.2 Classification of D-branes

So far we have only analyzed case (a) in (4.1). We can analyze the other D-branes (b)–(f) in the same way as (4.3) using the transformations given in appendix A. We omit the details of this analysis and summarize the results. The D-branes for cases $(d), (e)$ and (f) seem to exist only when β is rational as will be discussed employing the boundary state analysis in the next subsection.

(a) D0-brane located on the fixed lines $r_1 r_2 = 0$. In the rational case, we can also obtain a D2-brane wrapped on the torus with gauge flux $2\pi\alpha' F_{\varphi_1\varphi_2} = \frac{1}{b}$.

(b) D4-brane filling the whole spacetime (no flux).

(c) D2-brane filling the plane $X_2, \bar{X}_2$ (no flux).

(d) D3-brane ($= S^1 \times \mathbb{C}$; r_1 =fixed) with flux $2\pi\alpha' F_{\varphi_1\varphi_2} = \frac{1}{b}$. This brane exists only in the rational case.

(e) D4-brane filling the whole spacetime with flux $2\pi\alpha' F_{\varphi_1\varphi_2} = \frac{1}{b}$. This brane exists only in the rational case.

(f) D3-brane (φ_1 = fixed, i.e. $\mathbb{R}^3$) with no flux. This brane exists only in the rational case.

4.3 Boundary states

To describe D-branes including α' corrections, it is useful to construct boundary states. At rational values of β, as we have seen in section 3, the theory is equivalent to the orbifold with discrete torsion $\mathbb{C}^2/\mathbb{Z}_N \times \mathbb{Z}_N$ (see also appendix C). In this case we can compare the D-brane spectrum with the expected result.

First, let us construct the boundary states $|B\rangle$ for $(a), (b)$ and (c). In these cases, we can show that the identity

$$(J_i + \tilde{J}_i)|B\rangle = J_{i0}|B\rangle\,, \qquad i = 1, 2\,, \tag{4.11}$$

holds, where J_{i0} denotes the zero mode part of the angular momentum $J_{i0} = i\sqrt{\frac{2}{\alpha'}}(x_{i0}\bar{\alpha}_{i0} - \bar{x}_{i0}\alpha_{i0})$. A similar identity holds in the Melvin background [14], and in this subsection we follow conventions of that paper. In other words, these boundary conditions (i.e. (purely) Neumann or Dirichlet in each X^i direction) ensure that the oscillator parts of the boundary states do not have any J_i charges.

In the (b) D-brane case, the zero mode parts J_{i0} are also zero due to the Neumann boundary condition. Thus its boundary state is the same as the familiar one in the flat space. At rational β, it is equivalent to a (bulk) D4-brane in the orbifold theory with discrete torsion.

The (a) D-brane boundary states have non-zero values of J_{i0} in general. It looks impossible to write down boundary states when J_{i0} is an arbitrary integer due to the self-interacting mode shifts (2.11). However, if we assume that either r_1 or r_2 is zero, we have $\hat{J}_{10} = 0$ or $\hat{J}_{20} = 0$. Then we can write down the boundary state (we consider the $r_1 = 0$

case and suppress fermions for simplicity)

$$|B\rangle = \sum_{J_{20}\in\mathbb{Z}} \exp\left[\sum_{n=0}^{\infty} \frac{1}{n+\nu_1}\alpha^1_{-n-\nu_1}\bar{\tilde{\alpha}}^1_{-n-\nu_1} + \frac{1}{n}\alpha^2_{-n}\bar{\tilde{\alpha}}^2_{-n} + h.c.\right] |J_{10}=0, J_{20}\rangle_{\text{zeromode}}\,, \tag{4.12}$$

up to a normalization factor which can be determined by open-closed duality. This D-brane has the moduli to move in the X^2 direction. The other one located at $r_2 = 0$ can be found in the same way. When β is rational, we can classify the values of J_{20} by the integers mod N $(m = 0, 1, 2, \cdots, N-1)$

$$|J_{20} = m\rangle_{\text{zeromode}} = \frac{1}{N}\sum_{l=0}^{N-1} e^{\frac{2\pi i}{N}(\hat{J}_2 - m)l}|X_2\rangle_{\text{zeromode}}\,. \tag{4.13}$$

Then the boundary state (4.12) agrees with the $\mathbb{C}^2$ analogue of the boundary state for a fractional D0-brane computed in [21] for the orbifold $\mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_N$. In the rational case we can also define a D0-brane which can freely move in any direction by projecting the angular momenta to $\hat{J}_{1,2} \in N\mathbb{Z}$. This can be done by acting with the operators $\Omega_i = \sum_{l=0}^{N-1} e^{2\pi i \hat{J}_i l/N}$ on the boundary states (4.12). Its boundary state only includes the untwisted sector and is equivalent to a bulk (or regular) D0-brane in the orbifold theory. This construction manifestly shows that the bulk D0-brane is made of N fractional D0-branes at N different positions.

The boundary state for a (c) D-brane can be constructed almost in the same way as the (a) case. This corresponds to the (bulk) D2-brane in the orbifold theory.

The analysis of boundary state for cases $(d), (e), (f)$ is a bit different from the previous ones. We no longer have the relation (4.11) due to the mixed ND boundary condition. However, we can still project the whole boundary states by the operators Ω_i when β is rational. Notice that now they act also on the massive oscillators. Putting this operator in the ND part, we can straightforwardly find the boundary states for $(d), (e)$ and (f). They are a respectively the bulk D1, D2 and D3-branes in the orbifold theory.

5. Three parameter model

In this section we consider the generalization of (1.1) to a three parameter deformation of $\mathbb{C}^3$. Similar to the AdS case studied in [1], we can make a sequence of transformations, $(TsT)_{b_1}(TsT)_{b_2}(TsT)_{b_3}$ of flat space to obtain the background [22]

$$\begin{aligned} ds^2 &= \sum_{i=1}^{3}(dr_i^2 + gr_i^2 d\phi_i) + gr_1^2r_2^2r_3^2(\sum_{i=1}^{3} b_i d\phi_i)^2, \\ B &= g(b_3 r_1^2 r_2^2 d\phi_1\wedge d\phi_2 + b_1 r_2^2 r_3^2 d\phi_2 \wedge d\phi_3 + b_2 r_3^2 r_1^2 d\phi_3\wedge d\phi_1)\,, \\ e^{2\phi} &= e^{2\phi_0} g\,, \qquad g^{-1} = 1 + b_3^2 r_1^2 r_2^2 + b_1^2 r_2^2 r_3^2 + b_2^2 r_3^2 r_1^2\,. \end{aligned} \tag{5.1}$$

Below we will use different angular variables related to ϕ_i by

$$\psi = \frac{1}{3}(\phi_1+\phi_2+\phi_3)\,, \qquad \varphi_1 = \frac{1}{3}(\phi_2+\phi_3-2\phi_1)\,, \qquad \varphi_2 = \frac{1}{3}(\phi_1+\phi_3-2\phi_2)\,. \tag{5.2}$$

The background preserves supersymmetry when $|b_1| = |b_2| = |b_3|$, as we will see shortly.

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5.1 Partition function

As shown in [8], the sigma model for closed strings in this background can be solved with the same kind of transformations we reviewed in section 2. In the canonical quantization, one has

$$\begin{aligned} L_0 &= N - a - \sum_{k=1}^{3} (\nu_k - [\nu_k]) J_k \,, \\ \tilde{L}_0 &= \tilde{N} - \tilde{a} + \sum_{k=1}^{3} (\nu_k - [\nu_k]) \tilde{J}_k \,, \end{aligned} \tag{5.3}$$

where

$$\nu_i = \epsilon_{ijk} (J_j + \tilde{J}_j) \beta_k \qquad (\beta_i \equiv b_i \alpha') \,. \tag{5.4}$$

We can immediately write down the analogue of (3.1) and (3.7),

$$Z_{T^2} = i V_{D-6} \int \frac{d\tau d\bar{\tau}}{4\tau_2} \frac{1}{(4\pi^2 \alpha' \tau_2)^{\frac{1}{2}(D-6)}} Z(\tau, \bar{\tau}, \beta) \,, \tag{5.5}$$

and

$$\begin{aligned} Z = & \int d^2 j_1 \, d^2 j_2 \, d^2 j_3 \, d^2 \chi_1 \, d^2 \chi_2 \, d^2 \chi_3 \, \times \\ & \times e^{-2\pi i (\chi_1 j_1 + \bar{\chi}_1 \bar{j}_1 + \chi_2 j_2 + \bar{\chi}_2 \bar{j}_2 + \chi_3 j_3 + \bar{\chi}_3 \bar{j}_3)} e^{2\pi i \tau [-\nu_1 j_1 - \nu_2 j_2 - \nu_3 j_3]} e^{-2\pi i \bar{\tau} [\nu_1 \bar{j}_1 + \nu_2 \bar{j}_2 + \nu_3 \bar{j}_3]} \times \\ & \times \left| \frac{\vartheta_1(\frac{1}{2}\chi_{+++}|\tau) \vartheta_1(\frac{1}{2}\chi_{++-}|\tau) \vartheta_1(\frac{1}{2}\chi_{+-+}|\tau) \vartheta_1(\frac{1}{2}\chi_{+--}|\tau)}{\eta(\tau)^3 \vartheta_1(\chi_1|\tau) \vartheta_1(\chi_2|\tau) \vartheta_1(\chi_3|\tau)} \right|^2 , \end{aligned} \tag{5.6}$$

where

$$\chi_{s_1 s_2 s_3} = s_1 \chi_1 + s_2 \chi_2 + s_3 \chi_3 \,. \tag{5.7}$$

Plugging in (5.4) and performing the j integrals gives

$$\begin{aligned} Z = & \int \frac{d^2 \chi_1 \, d^2 \chi_2 \, d^2 \chi_3}{(2\tau_2)^2} \delta^2(\beta_1 \chi_1 + \beta_2 \chi_2 + \beta_3 \chi_3, \beta_1 \bar{\chi}_1 + \beta_2 \bar{\chi}_2 + \beta_3 \bar{\chi}_3) \times \\ & \times e^{\frac{\pi}{\beta_3 \tau_2} (\chi_1 \bar{\chi}_2 - \chi_2 \bar{\chi}_1)} \left| \frac{\vartheta_1(\frac{1}{2}\chi_{+++}|\tau) \vartheta_1(\frac{1}{2}\chi_{++-}|\tau) \vartheta_1(\frac{1}{2}\chi_{+-+}|\tau) \vartheta_1(\frac{1}{2}\chi_{+--}|\tau)}{\eta(\tau)^3 \vartheta_1(\chi_1|\tau) \vartheta_1(\chi_2|\tau) \vartheta_1(\chi_3|\tau)} \right|^2 . \end{aligned} \tag{5.8}$$

Recalling that $\vartheta_1(0|\tau) = 0$, it is manifest that (5.8) vanishes when $|b_1| = |b_2| = |b_3|$, indicating the presence of unbroken spacetime supersymmetry. This is similar to the mechanism of [25, 28].

One can also derive this result from a path integral calculation using the world-sheet action

$$\begin{aligned} S = \frac{1}{2\pi\alpha'} \int d^2 z \Big[& \frac{1}{b_3} (\bar{u} v - \bar{v} u) + \frac{1}{2} \left[(\partial + iu) Z_1 \cdot (\bar{\partial} - i\bar{u}) \bar{Z}_1 + (\bar{\partial} + i\bar{u}) Z_1 \cdot (\partial - iu) \bar{Z}_1 \right] + \\ & + \frac{1}{2} \left[(\partial + iv) Z_2 \cdot (\bar{\partial} - i\bar{v}) \bar{Z}_2 + (\bar{\partial} + i\bar{v}) Z_2 \cdot (\partial - iv) \bar{Z}_2 \right] + \\ & + \frac{1}{2} \Big[\left(\partial - i \frac{(b_1 u + b_2 v)}{b_3} \right) Z_3 \cdot \left(\bar{\partial} + i \frac{(b_1 \bar{u} + b_2 \bar{v})}{b_3} \right) \bar{Z}_3 + \\ & + \left(\bar{\partial} - i \frac{(b_1 \bar{u} + b_2 \bar{v})}{b_3} \right) Z_3 \cdot \left(\partial + i \frac{(b_1 u + b_2 v)}{b_3} \right) \bar{Z}_3 \Big] \Big] . \end{aligned} \tag{5.9}$$

It is straightforward to check that (5.9) becomes the same sigma model defined from the three parameter model (5.7) after integrating out u and v, and that its one-loop functional determinant agrees with (5.8).

We can study the properties of the partition function (5.8) as in section 3. Consider the supersymmetric case $\beta_1 = \beta_2 = \beta_3 (\equiv \beta)$. Then we can show the (formal) periodicity $Z(\beta+1, \tau, \bar{\tau}) = Z(\beta, \tau, \bar{\tau})$ in the same way as in (3.12). In the rational case $\beta = k/N$ we find that the partition function is equivalent to that of the orbifold $\mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_N$ with discrete torsion. The orbifold actions are defined by

$$\begin{aligned} g_1 &: (X_1, X_2, X_3) \rightarrow (e^{2\pi i k/N} X_1, X_2, e^{-2\pi i k/N} X_3)\,, \\ g_2 &: (X_1, X_2, X_3) \rightarrow (X_1, e^{2\pi i k/N} X_2, e^{-2\pi i k/N} X_3)\,. \end{aligned} \tag{5.10}$$

5.2 D-branes

The analysis of D-branes in this background can be done as in section 4. We continue to consider only the supersymmetric case $\beta_1 = \beta_2 = \beta_3 (= \beta)$ and only discuss the supersymmetric D0-brane in the free field description (the analogue of case (a) from section 4). In this example, the Dirichlet condition $(\partial - \bar{\partial})\varphi_1' = (\partial - \bar{\partial})\varphi_2' = (\partial - \bar{\partial})\psi = 0$ can be rewritten using the formula (A.6) into

$$\begin{aligned} (r_1^2 + r_3^2)(\partial + \bar{\partial})\varphi_1 + (r_3^2 - r_1^2)(\partial + \bar{\partial})\psi + r_3^3(\partial + \bar{\partial})\varphi_2 - b^{-1}(\partial - \bar{\partial})\varphi_2 &= 0\,, \\ (r_2^2 + r_3^2)(\partial + \bar{\partial})\varphi_2 + (r_3^2 - r_2^2)(\partial + \bar{\partial})\psi + r_3^3(\partial + \bar{\partial})\varphi_1 + b^{-1}(\partial - \bar{\partial})\varphi_1 &= 0\,, \\ (\partial - \bar{\partial})\psi &= 0\,. \end{aligned} \tag{5.11}$$

Comparing (5.11) with the standard formula as before we find a D2-brane wrapped on the torus (φ_1, φ_2) at any fixed values of ψ, r_1, r_2 and r_3 with gauge flux $2\pi\alpha' F_{\varphi_1,\varphi_2} = 1/b$. Again the rational values $\beta (= \alpha' b) = k/N$ are required for flux quantization. This flux together with the background B-field makes its world-volume a non-commutative torus with non-commutativity $2\pi\beta$ (see also appendix C). This fact nicely agrees with the β deformation of $\mathcal{N} = 4$ Yang-Mills theory [1, 10, 18] realized on D5-branes wrapped on the torus.

6. Time-dependent background via β deformation

We can also study β deformed $\mathbb{R}^{1,3}$ by replacing one of the two complex free fields $(X^1, \bar{X}^1)$ in (2.1) with a lorentzian one (X^+, X^-) with metric $ds^2 = -dX^+ dX^-$. We leave $(X^2, \bar{X}^2)$ unchanged. Define polar coordinates by $X^+ = te^{\theta'}$, $X^- = te^{-\theta'}$ and $X^2 = re^{i\phi}$ $(-\infty < t < \infty, 0 \le r \le \infty)$. These coordinates cover one half of the four dimensional Minkowski spacetime $\mathbb{R}^{1,3}$.

Now we perform T-duality in the θ' direction, shift by $\phi = \varphi_2 + b\tilde{\theta}$, and T-dualize in the $\tilde{\theta}$ direction again. Then we get the following time-dependent background[5]

$$ds^2 = -dt^2 + dr^2 + \frac{t^2}{1 + b^2 t^2 r^2} d\theta^2 + \frac{r^2}{1 + b^2 t^2 r^2} d\varphi^2\,,$$

[5] This can also be obtained from the analytic continuation $t \to it$, $\theta \to i\theta$ and $b \to ib$ of (1.1).

$$\begin{aligned} e^{2(\phi-\phi_0)} &= \frac{1}{1+b^2t^2r^2}\,, \\ B_{\theta\varphi} &= -\frac{bt^2r^2}{1+b^2t^2r^2}\,. \end{aligned} \tag{6.1}$$

The angular coordinate φ is periodic $\varphi \sim \varphi + 2\pi$ as usual. The boost coordinate θ can be compact or non-compact. Below we assume that θ is compact with $\theta \sim \theta + 2\pi$ and also that $\beta = \alpha' b$ is rational ($=k/N$). In the trivial case $\beta = 0$, the spacetime is the product of the Milne orbifold $[\mathbb{R}^{1,1}/\mathbb{Z}]_{\Delta=2\pi}$ (i.e. the orbifold by the boost $\Delta = 2\pi$) and the flat space $\mathbb{C}$. For recent development of orbifold theoretic approaches to time-dependent backgrounds see [29] and references therein.

When $\beta \neq 0$ the spacetime shows an intriguing time-evolution. At $t = -\infty$, the spacetime is given by the orbifold $[\mathbb{R}^{1,1}/\mathbb{Z}]_{\Delta=2\pi/N} \times [\mathbb{C}/\mathbb{Z}_N]$. As time passes, a bubble of the new geometry which looks like $[\mathbb{R}^{1,1}/\mathbb{Z}]_{\Delta=2\pi} \times \mathbb{C}$ is created at the origin $r = 0$ and its radius expands as $r \sim \frac{1}{b|t|}$. At $t = 0$ it covers the whole region completely and thus the spacetime becomes $[\mathbb{R}^{1,1}/\mathbb{Z}]_{\Delta=2\pi} \times \mathbb{C}$. After that, the time-reversal process occurs and the spacetime goes back to the original orbifold.

This time evolution can be regarded as a time-dependent process of resolving an orbifold space and is very similar to the scenario[6] in localized closed string tachyon condensation on the non-supersymmetric orbifold $\mathbb{C}/\mathbb{Z}_N$ [12]. It is expected that this time-dependent model can be exactly solvable, as in the space-like case. We leave the detailed study of this model for future work.

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A. T-duality relations

Here we summarize the on-shell relations between the various fields employed in section 2, using the rule (2.5). From (2.1) to (2.2) we find

$$\begin{aligned} \partial\tilde{\varphi}_1 &= -\frac{br_1^2r_2^2}{1+b^2r_1^2r_2^2}\partial\varphi_2 - \frac{r_1^2}{1+b^2r_1^2r_2^2}\partial\varphi_1\,, \\ \bar{\partial}\tilde{\varphi}_1 &= -\frac{br_1^2r_2^2}{1+b^2r_1^2r_2^2}\bar{\partial}\varphi_2 + \frac{r_1^2}{1+b^2r_1^2r_2^2}\bar{\partial}\varphi_1\,. \end{aligned} \tag{A.1}$$

Then from (2.2) to (2.3) we find

$$\begin{aligned} \partial\phi_1 &= -\frac{1}{r_1^2}\partial\tilde{\varphi}_1\,, \\ \bar{\partial}\phi_1 &= \frac{1}{r_1^2}\bar{\partial}\tilde{\varphi}_1\,. \end{aligned} \tag{A.2}$$

[6] For time-dependent supergravity solutions for the decay of $\mathbb{C}/\mathbb{Z}_N$ refer to [30, 31].

Combining these results, we see that in going from (2.1) to (2.3) we have the transformation

$$\begin{aligned}\partial\phi_1 &= \frac{br_2^2}{1+b^2r_1^2r_2^2}\partial\varphi_2 + \frac{1}{1+b^2r_1^2r_2^2}\partial\varphi_1\,,\\ \bar\partial\phi_1 &= -\frac{br_2^2}{1+b^2r_1^2r_2^2}\bar\partial\varphi_2 + \frac{1}{1+b^2r_1^2r_2^2}\bar\partial\varphi_1\,.\end{aligned} \tag{A.3}$$

In the same way we find the following three relations for $\tilde\varphi_2$, which was defined in section 2 in a manner similar to $\tilde\varphi_1$:

$$\begin{aligned}\partial\tilde\varphi_2 &= \frac{br_1^2r_2^2}{1+b^2r_1^2r_2^2}\partial\varphi_1 - \frac{r_2^2}{1+b^2r_1^2r_2^2}\partial\varphi_2\,,\\ \bar\partial\tilde\varphi_2 &= \frac{br_1^2r_2^2}{1+b^2r_1^2r_2^2}\bar\partial\varphi_1 + \frac{r_2^2}{1+b^2r_1^2r_2^2}\bar\partial\varphi_2\,,\\ \partial\phi_2 &= -\frac{1}{r_2^2}\partial\tilde\varphi_2\,,\\ \bar\partial\phi_2 &= \frac{1}{r_2^2}\bar\partial\tilde\varphi_2\,,\\ \partial\phi_2 &= -\frac{br_1^2}{1+b^2r_1^2r_2^2}\partial\varphi_1 + \frac{1}{1+b^2r_1^2r_2^2}\partial\varphi_2\,,\\ \bar\partial\phi_2 &= \frac{br_1^2}{1+b^2r_1^2r_2^2}\bar\partial\varphi_1 + \frac{1}{1+b^2r_1^2r_2^2}\bar\partial\varphi_2\,.\end{aligned} \tag{A.4}$$

From the above relations we can show that

$$\partial\phi_1 = \partial(\varphi_1 - b\tilde\varphi_2)\,, \qquad \bar\partial\phi_1 = \bar\partial(\varphi_1 - b\tilde\varphi_2)\,. \tag{A.5}$$

Thus $\phi_1 = \varphi_1 - b\tilde\varphi_2$. Also we have $\phi_2 = \varphi_2 + b\tilde\varphi_1$ by definition.

For the supersymmetric ($b_i = b$ for $i = 1,2,3$) model considered in section 5 we find

$$\begin{aligned}\partial\varphi_1' &= \frac{1+br_3^2}{1+b^2g_0}\partial\varphi_1 + \frac{b(r_3^2+r_2^2)}{1+b^2g_0}\partial\varphi_2 + \frac{b^2[r_1^2r_2^2+r_1^2r_3^2-2r_2^2r_3^2]+b(r_3^2-r_2^2)}{1+b^2g_0}\partial\psi\,,\\ \bar\partial\varphi_1' &= \frac{1-br_3^2}{1+b^2g_0}\bar\partial\varphi_1 - \frac{b(r_3^2+r_2^2)}{1+b^2g_0}\bar\partial\varphi_2 + \frac{b^2[r_1^2r_2^2+r_1^2r_3^2-2r_2^2r_3^2]-b(r_3^2-r_2^2)}{1+b^2g_0}\bar\partial\psi\,,\\ \partial\varphi_2' &= \frac{1-br_3^2}{1+b^2g_0}\partial\varphi_2 - \frac{b(r_3^2+r_1^2)}{1+b^2g_0}\partial\varphi_1 + \frac{b^2[r_1^2r_2^2+r_1^2r_3^2-2r_2^2r_3^2]-b(r_3^2-r_1^2)}{1+b^2g_0}\partial\psi\,,\\ \bar\partial\varphi_2' &= \frac{1+br_3^2}{1+b^2g_0}\bar\partial\varphi_2 + \frac{b(r_3^2+r_1^2)}{1+b^2g_0}\bar\partial\varphi_1 + \frac{b^2[r_1^2r_2^2+r_1^2r_3^2-2r_2^2r_3^2]+b(r_3^2-r_1^2)}{1+b^2g_0}\bar\partial\psi\,.\end{aligned} \tag{A.6}$$

B. Partition functions

We record here some formulas which are useful for computing the partition functions appearing in section 3. We use the notation

$$q = e^{2\pi i\tau}\,, \qquad y = e^{2\pi i a}\,, \qquad z = e^{2\pi i b}\,. \tag{B.1}$$

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Consider first the bosonic oscillators. In D spacetime dimensions there are a total of $D-2$ physical bosons, with four of them describing the sigma model (2.1) and the remaining $D-6$ free. The partition function in the bosonic Hilbert space weighted by the angular momenta J_1 and J_2 is

$$\begin{aligned} Z_{\mathrm{B}} &\equiv \mathrm{Tr}_{\mathrm{B}}[q^N y^{J_1} z^{J_2}] \\ &= \frac{y^{-\frac{1}{2}}}{1-1/y}\frac{z^{-\frac{1}{2}}}{1-1/z}\prod_{n=1}^{\infty}\frac{1}{(1-q^n)^{D-6}}\frac{1}{1-q^n y}\frac{1}{1-q^n/y}\frac{1}{1-q^n z}\frac{1}{1-q^n/z} \\ &= -\frac{q^{(D-2)/24}\eta(\tau)^{8-D}}{\vartheta_1(a|\tau)\vartheta_1(b|\tau)}\,. \end{aligned} \tag{B.2}$$

The corresponding partition function for fermions in the Ramond sector is

$$\begin{aligned} Z_{\mathrm{R}} &\equiv \mathrm{Tr}_{\mathrm{R}}[q^N y^{J_1} z^{J_2}] \\ &= 4(y^{\frac{1}{2}}+y^{-\frac{1}{2}})(z^{\frac{1}{2}}+z^{-\frac{1}{2}})\prod_{n=1}^{\infty}(1+q^n)^4(1+q^n y)(1+q^n/y)(1+q^n z)(1+q^n/z) \\ &= q^{-1/3}\frac{\vartheta_2(0|\tau)^2\vartheta_2(a|\tau)\vartheta_2(b|\tau)}{\eta(\tau)^4}\,. \end{aligned} \tag{B.3}$$

The projection onto states of definite fermion number $e^{\pi i F}=\pm 1$ gives, for either sign, just a factor of 1/2,

$$Z_{\mathrm{R}\pm} = \frac{1}{2}Z_{\mathrm{R}}\,. \tag{B.4}$$

For the NS sector, we have

$$\begin{aligned} Z_{\mathrm{NS}} &\equiv q^{-\frac{1}{2}}\,\mathrm{Tr}_{\mathrm{NS}}[q^N y^{J_1} z^{J_2}] \\ &= q^{-\frac{1}{2}}\prod_{n=1}^{\infty}(1+q^{n-\frac{1}{2}})^4(1+q^{n-\frac{1}{2}}y)(1+q^{n-\frac{1}{2}}/y)(1+q^{n-\frac{1}{2}}z)(1+q^{n-\frac{1}{2}}/z) \\ &= q^{-1/3}\frac{\vartheta_3(0|\tau)^2\vartheta_3(a|\tau)\vartheta_3(b|\tau)}{\eta(\tau)^4}\,. \end{aligned} \tag{B.5}$$

Note that we have already included here the ground state energy $a=-1/2$, which accounts for the prefactor $q^{-1/2}$ in the first line. The projection onto definite fermion number $e^{\pi i F}=\pm 1$ now gives

$$Z_{\mathrm{NS}\pm} = \frac{q^{-1/3}}{2\eta(\tau)^4}\left[\vartheta_3(0|\tau)^2\vartheta_3(a|\tau)\vartheta_3(b|\tau) \mp \vartheta_4(0|\tau)^2\vartheta_4(a|\tau)\vartheta_4(b|\tau)\right]. \tag{B.6}$$

The type II GSO projection is given by

$$\begin{aligned} Z_{\mathrm{NS}+} - Z_{\mathrm{R}\pm} &= \frac{q^{-1/3}}{2\eta(\tau)^4}\sum_{i=2}^{4}(-1)^{i+1}\vartheta_i(0|\tau)^2\vartheta_i(a|\tau)\vartheta_i(b|\tau) \\ &= \frac{q^{-1/3}}{\eta(\tau)^4}\vartheta_1(\tfrac{1}{2}(a+b)|\tau)^2\vartheta_1(\tfrac{1}{2}(a-b)|\tau)^2\,, \end{aligned} \tag{B.7}$$

using Jacobi's abstruse identity. The $\pm$ subscript on Z_{R} indicates type IIA/IIB, which are equivalent in this context.

C. D-branes on orbifolds with discrete torsion

In orbifolds with discrete torsion, we can analyze D-branes by following the general procedures found in [16, 17]. Let the action of $g \in G$ on the Chan-Paton factor be denoted by $\gamma(g)$. The projection in the open string theory is given by

$$\phi = \gamma(g)^{-1} \cdot (g \cdot \phi) \cdot \gamma(g) \,, \tag{C.1}$$

as in the usual orbifold without discrete torsion [32]. In the presence of discrete torsion, $\gamma(g)$ provides a projective representation of G. In particular, the consistency of open and closed string theory gives the relation [16, 17, 19]

$$\gamma(g)\gamma(h) = \epsilon(g,h)\gamma(h)\gamma(g) \,. \tag{C.2}$$

Let us now recall the result we found for the case $\beta = 4/N$, which was given by (3.30) with $n = 1$. In this case (C.2) becomes the fuzzy torus algebra generated by $U = \gamma(1,0)$ and $V = \gamma(0,1)$, which satisfy $UV = VUe^{2\pi i/N}$. This has the standard $N \times N$ matrix representation. After imposing the orbifold projection (C.1), we find that the two transverse scalars Φ_1 and Φ_2 should be proportional to V^2 and U^{-2}, respectively. D-branes defined in this way are called fractional D-branes [33, 16, 17, 34, 19]. Note that an $N \times N$ Chan-Paton matrix is employed to describe a fractional D-brane. For its boundary state refer to [21].

Let us consider fractional D0-branes. Since the twisted sectors for g_1 (or g_2) are included, there are no zero-modes for X_1 (or X_2) and they cannot freely move in those directions. The non-abelian gauge theory on N such D0-branes can be realized by multiplying another Chan-Paton degree of freedom with this $N \times N$ matrix. When we make the latter matrix implicit, we will have a non-commutative deformation. The commutator $[\Phi_1, \Phi_2] = \Phi_1\Phi_2 - \Phi_2\Phi_1$ is β deformed as

$$[\Phi_1, \Phi_2]_\beta = e^{\pi i \beta}\Phi_1\Phi_2 - e^{-\pi i\beta}\Phi_2\Phi_1 \,, \tag{C.3}$$

where $\beta = 4/N$ as expected. It is natural to believe this deformation continues to be true when β is irrational, in which case the boundary state analysis shows that there also exist fractional D0-branes. When the Chan-Paton matrix has size N and Φ_1 and Φ_2 are respectively proportional to V^2 and U^2, we see that they have vanishing β deformed commutator. This represents a regular (or bulk) D0-brane which can move freely in any directions.

Even though we analyzed D-branes in the particular case $\mathbb{C}^2/\mathbb{Z}_N \times \mathbb{Z}_N$, we can obtain almost the same results for the supersymmetric orbifold $\mathbb{C}^3/\mathbb{Z}_N \times \mathbb{Z}_N$ [18]. This is equivalent to the three parameter model we study in section 5 at $\beta_i = 1/N$.

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